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Site bond percolation in ferromagnetic and antiferromagnetic Ising models: a renormalisation group approach

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Abstract. The Migdal–Kadanoff infinitesimal renormalisation group is employed to study the site bond correlated percolation problem, in a ferromagnetic and antiferromagnetic lattice gas. The general phase diagram shows that percolation is favoured by ferromagnetic interaction, while inhibited by antiferromagnetic interaction. Moreover, the lattice gas critical point is never a percolation point in the antiferromagnetic case, while it is a percolation point in the ferromagnetic case for a particular range of bond probability p_B.

1. Introduction

The concept of clusters has been widely used in many fields of physics particularly in connection with critical phenomena (Binder 1976). In the site percolation problem the clusters are defined as the maximal set of nearest neighbour particles distributed on a lattice. In standard percolation the particles are randomly distributed while in correlated percolation they interact (see the review articles by Stauffer (1979) and Essam (1980)). The case in which the particles are correlated with ferromagnetic interaction is the most studied, for example, the lattice gas or Ising model. More recently a generalisation of this problem has been proposed: the site bond correlated percolation problem in which the clusters in a lattice gas are defined as the maximal sets of nearest neighbour particles connected by active bonds. The probability of a bond being active is p_B and non-active $1-p_B$. This model has been useful for describing the effect of bad solvent in gelation (Coniglio *et al* 1979). It also provides a useful description for the droplets in the lattice gas or Ising model (Coniglio and Klein 1980). Very little attention has been devoted to the correlated percolation problem in which the correlation is antiferromagnetic (Stoll and Domb 1979, Murata 1979, Napiorkowski and Hemmer 1980). In this paper we want to study the site bond correlated percolation problem for both ferromagnetic and antiferromagnetic interactions on the square lattice. We follow closely the formalism developed by Coniglio and Klein for the ferromagnetic site bond correlated percolation. They use a Hamiltonian formalism to which they apply the finite Migdal–Kadanoff renormalisation group (MKRG) (Migdal 1976, Kadanoff 1976). Since the finite renormalisation group employed is not suitable for treating the antiferromagnetic case, we have used the infinitesimal renormalisation group which

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gives better results for the ferromagnetic case and is also suitable for describing the antiferromagnetic case. In § 2 we briefly review the properties of the lattice gas Q-state Potts model and show that in the $Q = 1$ limit it gives the site bond correlated percolation problem. In § 4 we apply the infinitesimal MKRG to both the ferromagnetic and antiferromagnetic case and draw the conclusions. In the appendix details of the MKRG are reported.

2. Site bond correlated percolation as $Q = 1$ limit of the Q-state lattice gas Potts model

Consider the lattice gas Hamiltonian $-\beta\mathcal{H}_{\mathrm{LG}}$ on a regular lattice of N sites

$$-\beta\mathcal{H}_{\mathrm{LG}} = K \sum_{\langle i,j \rangle} n_i n_j - \Delta \sum_i n_i \tag{1}$$

where $n_i = 1$ if site i is occupied, 0 otherwise. $\beta = 1/K_{\mathrm{B}}T$, K is the nearest neighbour coupling constant related to the Ising coupling constant K_I by $K_I = \frac{1}{4}K$, Δ is the chemical potential related to the Ising magnetic field H and the coordination number c by $H = \frac{1}{2}(\Delta - \frac{1}{2}cK)$. The sum is over nearest neighbours. Positive and negative values of K correspond to ferromagnetic and antiferromagnetic interactions, respectively. In site bond correlated percolation a cluster is defined as the maximal set of nearest neighbour particles connected by active bonds. The probability of a bond being active is p_{B} and non-active $1 - p_{\mathrm{B}}$. The bonds are only introduced to define the connectivity between two nearest neighbour particles and do not affect their interacting energy and, therefore, the particle distribution. Every particle configuration $\{n_i\}$ is weighted by the Boltzmann factor $\exp(-\beta\mathcal{H}_{\mathrm{LG}})$.

The quantities of interest are: the average number of clusters of s particles per site $\langle n_s \rangle$, the average number of clusters per site

$$\langle N_{\mathrm{cl}} \rangle = \sum_s \langle n_s \rangle, \tag{2}$$

the percolation probability

$$P = 1 - \Sigma' s \langle n_s \rangle \rho^{-1} \tag{3}$$

(ρ is the density of particles), the mean cluster size

$$S = (\Sigma' s^2 \langle n_s \rangle)(\Sigma' s n_s)^{-1}, \tag{4}$$

the pair connectedness function

$$\pi_{ij} = \langle \gamma_{ij} \rangle \tag{5}$$

where γ_{ij} is 1 if sites i and j belong to the same finite cluster, 0 otherwise. Σ' is the sum over all finite clusters.

Here the brackets $\langle \ldots \rangle$ stand for the average over the sites and bonds

$$\langle \ldots \rangle = \lim_{N \to \infty} \sum_{\{n_i\}} \exp(-\beta\mathcal{H}_{\mathrm{LG}}) \Big(\sum_{C \subseteq E\{n_i\}} \ldots p_{\mathrm{B}}^{|C|} |1 - p_{\mathrm{B}}|^{|D|} \Big) \Big(\sum_{\{n_i\}} \exp(-\beta\mathcal{H}_{\mathrm{LG}}) \Big)^{-1} \tag{6}$$

where $E\{n_i\}$ is the set of all bonds in the sublattice made of the occupied sites in the configuration $\{n_i\}$. C is a subset of $E\{n_i\}$ and $D = E\{n_i\} - C$; $|C|$ and $|D|$ are the number of bonds respectively in the subsets C and D. Note that $\Sigma_{C \subseteq E\{n_i\}} p_{\mathrm{B}}^{|C|} |1 - p_{\mathrm{B}}|^{|D|} = 1$ for

every configuration $\{n_i\}$. It is well known (Kastelyn and Fortuin 1969) that the random bond percolation problem can be obtained from the Q-state Potts model in the $Q=1$ limit.

Analogously the site bond correlated percolation can be obtained from the lattice gas Potts model in the $Q=1$ limit (Murata 1979, Coniglio and Klein 1980). For convenience we give the derivation here. We start from the lattice gas Q-state Potts model $-\beta\mathcal{H}$

$$-\beta\mathcal{H} = J\sum_{\langle ij\rangle}(\delta_{\sigma_i\sigma_j}-1)n_in_j + h\sum_i(\delta_{\sigma_i 1}-1)n_i - \beta\mathcal{H}_{\mathrm{LG}} \tag{7}$$

where $\sigma_i = 1 \ldots Q$ are the Q-state Potts variables. n_i are the lattice gas variables and $\beta\mathcal{H}_{\mathrm{LG}}$ is the lattice gas Hamiltonian defined in (1). Hamiltonian (7) has recently received considerable attention (Nienhuis *et al* 1979) due to the role played by the vacancies in a real space renormalisation group of the Q-state Potts model. Starting from (7) we can write the partition function Z in the following way

$$Z = \sum_{\{n_i\}}\exp(-\beta\mathcal{H}_{\mathrm{LG}})\sum_{\{\sigma_i\}}\prod_{\langle ij\rangle}(q_{ij}+p_{ij}\,\delta_{\sigma_i\sigma_j})\prod_i\{\exp(-hn_i)+[1-\exp(-hn_i)]\,\delta_{\sigma_i 1}\} \tag{8}$$

where

$$q_{ij} = \exp(-Jn_in_j) \qquad p_{ij} = 1-q_{ij} \tag{9}$$

$$Z = \sum_{\{n_i\}}\exp(-\beta\mathcal{H}_{\mathrm{LG}})Q^{\Sigma_i(1-n_i)}Z_{\mathrm{Potts}}\{n_i\} \tag{10}$$

where

$$Z_{\mathrm{Potts}}\{n_i\} = \mathrm{Tr}\sum_{\{\sigma_i\}}\sum_{C\subseteq E\{n_i\}}q^{|D|}p^{|C|}\prod_{\langle ij\rangle\in C}\delta_{\sigma_i\sigma_j}\prod_{i\in\theta\{n_i\}}[\mathrm{e}^{-h}+(1-\mathrm{e}^{-h})\,\delta_{\sigma_i 1}] \tag{11}$$

is the partition function of the Q-state Potts model defined on the sublattice made of the occupied sites in the configuration $\{n_i\}$. $\theta\{n_i\}$ is the set of vertices in this sublattice and $E\{n_i\}$ the set of bonds; $q=\mathrm{e}^{-J}$, $p=1-q$. In (10) the term $Q^{\Sigma_i(1-n_i)}$ is due to the trace over all the unoccupied sites where $n_i=0$. Following the usual procedure as for the regular lattice (Wu 1978, Murata 1979)

$$Z_{\mathrm{Potts}}\{n_i\} = \sum_{C\subseteq E\{n_i\}}p^{|C|}q^{|D|}\prod_r[(Q-1)\exp(-hs_r)+1] \tag{12}$$

where r labels the clusters in configuration C, s_r is the number of sites in the rth cluster and the product is over all the clusters in the configuration C. Equations (10) and (12) in the $Q=1$ limit give rise to the same probability distribution as in (6) provided that $p_{\mathrm{B}}=p=1-\mathrm{e}^{-J}$.

Let us define the 'free energy' F in the following way

$$F = \lim_{N\to\infty}\left[\frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}Q}\ln Z\right]_{Q=1}. \tag{13}$$

From equation (12) we have

$$F = (1-\rho)+\langle N_{\mathrm{cl}}\rangle \tag{14}$$

$$\left.\frac{\mathrm{d}F}{\mathrm{d}h}\right|_{h=0} = \sum_s s\langle n_s\rangle \tag{15}$$

$$\left.\frac{\mathrm{d}^2F}{\mathrm{d}h^2}\right|_{h=0} = \sum_s s^2\langle n_s\rangle \tag{16}$$

which are related to equations (2), (3) and (4).

Analogously the pair connectedness π_{ij} can be obtained by introducing inhomogeneous fields h_i at each site i

$$\pi_{ij} = \left.\frac{\partial^2 F}{\partial h_i\,\partial h_j}\right|_{h_i=h_j=0}. \tag{17}$$

3. Ising clusters and critical droplets

In the previous section we have shown that all the connectivity properties can be obtained from the lattice gas Q-state Potts model (7) in the $Q=1$ limit.

We shall show now how Hamiltonian (7) is equivalent to an asymmetric $(Q+1)$-state Potts model (Berker *et al* 1978).

Define a new $(Q+1)$-valued variable b_i related to the (σ, n) variables by the following transformations

$$b_i = \begin{cases} 1\ldots Q & \text{if } n_i = 1 \text{ and } \sigma_i = 1\ldots Q \\ 0 & \text{if } n_i = 0. \end{cases}$$

In terms of this new variable Hamiltonian (7) with zero ghost field $h=0$ can be written

$$-\beta\bar{\mathcal{H}}\{b_i\} = J\sum_{\langle ij\rangle}(\delta_{b_ib_j}-1) - (2J-K)\sum_{\langle ij\rangle}\delta_{b_i0}\,\delta_{b_j0} + [\Delta + C(J-K)]\,\delta_{b_i0} + \tfrac{1}{2}CNK - \Delta N \tag{18}$$

and the partition function Z can be expressed as

$$Z = \sum_{\{n_i,\sigma_i\}}\exp(-\beta\bar{\mathcal{H}}\{n_i,\sigma_i\})$$

$$= \sum_{\{b_i\}}\exp\Big(\ln Q\sum_i \delta_{b_i,0} - \beta\bar{\mathcal{H}}\{b_i\}\Big). \tag{19}$$

The term $\exp(\ln Q\,\Sigma_i\,\delta_{b_i0})$ comes from the trace over all the σ_i variables at the sites where $n_i=0$. In conclusion Hamiltonian (7) with $h=0$ written in terms of the new variables $\{b_i\}$ is equivalent to the following asymmetric $(Q+1)$-state Potts model:

$$-\beta\bar{\mathcal{H}}\{b_i\} = J\sum_{ij}(\delta_{b_ib_j}-1) - 2(J-\tfrac{1}{2}K)\sum_{ij}\delta_{b_i0}\,\delta_{b_j0}$$

$$+[2H + C(J-\tfrac{1}{2}K) + \ln Q]\sum_i \delta_{b_i0} \tag{20}$$

where the Ising external field $H = \frac{1}{2}(\Delta - \frac{1}{2}CK)$ has been used. Therefore the site bond correlated percolation problem is obtained from the Hamiltonian (20) in the $Q=1$ limit. In this limit Hamiltonian (20) will tend to a two-state asymmetric Potts model.

Let us remember that K is the lattice gas coupling constant and J is related to the bond probability $p_B = 1-\mathrm{e}^{-J}$. For $J=\frac{1}{2}K$ i.e.

$$p_B = 1-\mathrm{e}^{-K/2} \tag{21}$$

Hamiltonian (20) in the $Q=1$ limit becomes a symmetric two-state Potts model with coupling constant $\frac{1}{2}K$, which is equivalent to a lattice gas with constant K.

Therefore the free energy has the usual lattice gas singularity at the critical point $H = 0$, $K = K_C$.

In conclusion this argument shows that in a lattice gas or Ising model the clusters made of nearest neighbour particles connected by active bonds with probability given by equation (21) diverge at the Ising critical point with Ising exponents. Namely the linear dimension ξ_p diverges with the Ising correlation exponent ν and the mean cluster size S with the Ising susceptibility exponent γ (Coniglio and Klein 1980) as in Fisher's (1967) droplet model. These new clusters will be called 'Ising droplets'. A Monte Carlo study of these droplets has been recently done by Stauffer (1981).

The importance of this definition lies in the fact that the usual clusters made of nearest neighbour particles, which we shall call 'Ising clusters', diverge in dimension higher than two at the wrong temperature (Müller-Krumbhaar 1974) while in two dimensions the Ising clusters diverge at the Ising critical temperature (Coniglio *et al* 1977) but the mean cluster size exponent is larger than the Ising susceptibility (Sykes and Gaunt 1976, Coniglio and Klein 1980).

We note that for the antiferromagnetic case, the asymmetric term in (20) can never vanish since $J > 0$ ($p_B > 0$). This suggests that the clusters made of nearest neighbour particles connected by bonds can never describe droplets in an antiferromagnetic lattice gas. We suggest that a cluster of holes and particles with antiferromagnetic order would be suitable as a definition of droplet.

4. Infinitesimal MKRG for ferromagnetic and antiferromagnetic lattice gas Potts model

Hamiltonian (7) has been studied extensively in the ferromagnetic case $K > 0$ (Berker *et al* 1978) using the Migdal–Kadanoff renormalisation group. Since this group is not suitable for treating the antiferromagnetic case, we have employed the infinitesimal MKRG which gives better results for the ferromagnetic case and is also suitable for describing the antiferromagnetic case.

In the appendix we give a general outline of MKRG. This method applied to the lattice gas Potts model (7), in the $Q = 1$ limit, for the ghost field $h = 0$ and for the ferromagnetic case gives the following recursion relations (cf Coniglio and Zia unpublished)

$$y' = \tfrac{1}{4}(\beta^2 - \alpha^2) \tag{22a}$$

$$w' = (\alpha + \beta)/(\beta - \alpha) \tag{22b}$$

$$x' = \left(1 - \frac{2\beta}{\alpha + \beta}\gamma\right)^{-1} \tag{22c}$$

where $y = e^K$, $w = e^{-H}$, $x = e^J$, $y' = e^{K'}$, $w' = e^{-H'}$, $x' = e^{J'}$ and

$$\alpha = \frac{w^b - 1}{w^{b/2}} y^{b/2} \qquad \beta = \sqrt{\alpha^2 + 4}\,\frac{G+1}{G-1}$$

$$\eta = \frac{w^b + 1}{w^{b/2}} y^{b/2} \qquad G = \tilde{G}^b = \left(\frac{\eta + \sqrt{\alpha^2+4}}{\eta - \sqrt{\alpha^2+4}}\right)^b$$

$$\gamma = \left(2\frac{x^b - 1}{x^b}\frac{w^b}{w^b + 1}\right)^b \frac{1}{[1+R]^b + [1-R]^b}$$

$$R = \sqrt{(w^b - 1)^2 + 4w^b y^{-b}}/w^b + 1.$$

In the antiferromagnetic case $K < 0$ we have $1 - R < 0$ and $\tilde{G} < 0$; therefore

$$(1-R)^b = (-1)^b |1-R|^b \qquad \text{and} \qquad G^b = (-1)^b |\tilde{G}|^b \tag{23}$$

for any integer value of b.

To continue to values of b infinitesimally near to 1, we replace relation (23) by

$$(1-R)^b = \cos b\pi |1-R|^b \qquad \text{and} \qquad G^b = \cos b\pi |\tilde{G}|^b.$$

We remark here that the renormalisation equations for y and w are decoupled from the x equation.

For b infinitesimally near to 1 the fixed points of relations (22) are given by solving the equations

$$\left.\frac{\partial y}{\partial b}\right|_{b=1} \equiv \varphi_1(y, w, x) = 0 \qquad \left.\frac{\partial w}{\partial b}\right|_{b=1} \equiv \varphi_2(y, w, x) = 0 \qquad \left.\frac{\partial x}{\partial b}\right|_{b=1} \equiv \varphi_3(y, w, x) = 0$$

and the scaling powers are obtained by solving the secular equation

$$\begin{vmatrix} \dfrac{\partial \varphi_1}{\partial y} - \lambda & \dfrac{\partial \varphi_1}{\partial w} & 0 \\ \dfrac{\partial \varphi_2}{\partial y} & \dfrac{\partial \varphi_2}{\partial w} - \lambda & 0 \\ \dfrac{\partial \varphi_3}{\partial y} & \dfrac{\partial \varphi_3}{\partial w} & \dfrac{\partial \varphi_3}{\partial x} - \lambda \end{vmatrix} = 0;$$

the zero elements are due to the decoupling of equations (22).

4.1. Ferromagnetic case

Equations (22*a*) and (22*b*) give the following fixed point

$$w = 1 \text{ i.e. } H = (\Delta - \tfrac{1}{2}CK) = 0$$

$$y = (\sqrt{2}+1)^2 \text{ i.e. } K_C = 4\,K_{ONS}$$

where

$$K_{ONS} = \tfrac{1}{2}\ln(\sqrt{2}+1) = 0.44069$$

and the trivial ones

$w = 1$	$y = \infty$		$H = 0$	$K = \infty$
$w = 1$	$y = 1$	i.e.	$H = 0$	$K = 0$
$w = \infty$	$y = 1$		$H = -\infty$	$K = 0.$

At the Onsager fixed point we obtain the following scaling powers

$$y_K = 0.75 \qquad y_H = 1.88.$$

The fixed point and flow lines for equations (22) are shown in figure 2. The fixed point $H = -\infty, K = 0, J_1 = 0.693$ (i.e. $p_B = \frac{1}{2}$) corresponds to pure bond percolation; this point determines the singular behaviour for all transitions which do not take place at the Ising

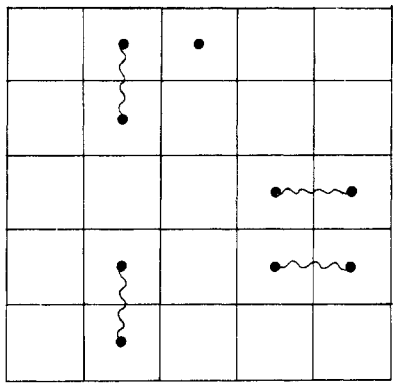

Figure 1. Occupied sites are denoted by dots and bonds by wavy lines. The configuration in this figure contains four two-site clusters and one one-site cluster.

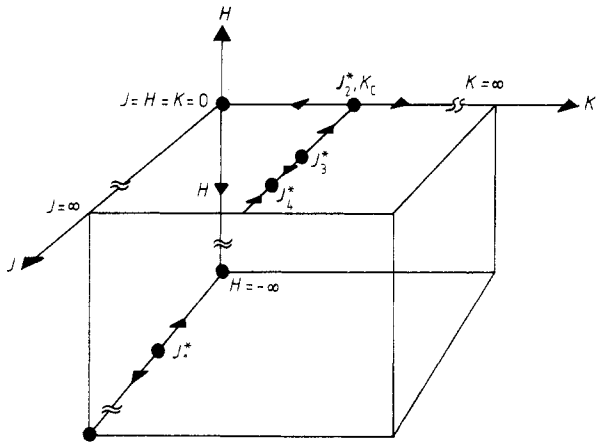

Figure 2. Schematic representation of the flow lines generated by the recursion relations of equations (22).

critical point $K = K_C$, $H = 0$. The scaling powers at this fixed point are $y_J = 0.61$ while y_K and y_H are negative this implies a connectedness length exponent $\nu_P = y_J^{-1} = 1.63$.

Let us consider the fixed points. $H = 0$, $K = K_C$: on this line there are three fixed points: $J_2 = 0$, $J_3 = \frac{1}{2}K_C = 2K_{ONS} = 0.88137$, $J_4 = 3.3068$. The fixed point J_4 is stable in the J direction. The scaling powers at J_3 and J_4 are respectively $y_J = 0.50$, $y_J = -1.43$. All these fixed points are characterised by the Ising scaling fields $K - K_C$ and H with the relative scaling powers $y_K = 0.75$, $y_H = 1.88$. At the J_2 fixed point we find $y_J < 0$. Therefore the only relevant scaling fields are $K - K_C$ and H. This fixed point describes the usual thermal Ising transition. At the fixed point J_3 we find $y_J = 0.50$. This is the most unstable fixed point and describes the critical behaviour of the Ising droplets, which diverge with Ising exponents (Coniglio and Klein 1980).

Finally at the J_4 fixed point we find $y_J < 0$. This point is stable with respect to J. It describes percolation at the Ising critical point with $1 - e^{-K/2} < p_B \leq 1$. In particular it describes the critical behaviour of the Ising cluster made of nearest neighbour particles ($p_B = 1$). Here the connectedness length $\xi_P = (K - K_C)^{-\nu}$, where $\nu = Y_K^{-1}$, has the same

singularity of the Ising correlation length exponent while the mean cluster size diverges with an exponent larger than the susceptibility (Sykes and Gaunt 1976, Coniglio and Klein 1980).

This behaviour is due to the sum of two contributions: one is due to geometrical effects, the other to correlations. Moreover, by looking at the flow lines we have been able to locate the critical lines of percolation points. In figure 3 are shown the critical lines of percolation points in the (H, T) plane for different values of $p_B = 1 - e^{-J}$. Each critical line in the (H, T) plane separates a percolating region (above the critical line), where p_B renormalises successively to 1, from a non-percolating region (below the critical line) where p_B renormalises successively to zero.

In figure 3 are plotted the critical lines for two values of p_B. Smaller values of p_B delimit smaller percolative regions.

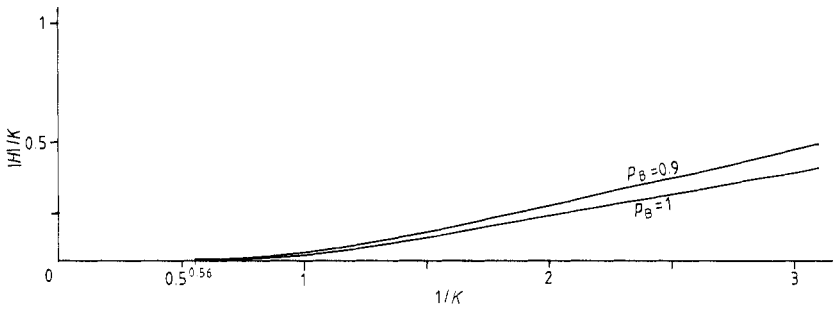

Figure 3. Percolation lines for different values of p_B in the ferromagnetic case. These lines end at the Ising critical point $H/K = 0$, $K^{-1} = K_C^{-1}$ for values of p_B satisfying the inequality $1 \leqslant p_B \leqslant 1 - e^{-K_C/2}$.

4.2. Antiferromagnetic case

As for the ferromagnetic case the recursion relations equations (22) for K and H are separated from J.

Equations (22*a*) and (22*b*) together with (23) give the antiferromagnetic Ising fixed point $H = 0, K = -K_C$ with the scaling power $y_K = 0.75$, $y_H = 0.118$. Moreover there is also another spurious fixed point near $H = 0$ which is due to the Migdal approximation (figure 4).

This spurious fixed point however does not affect the qualitative behaviour of the percolation line. In the plane (H, T), by looking at flow lines we find the antiferromagnetic Ising phase boundary (figure 4) in agreement with other results (di Liberto 1973, Racz 1980). The percolation properties are obtained from the overall study of equations (22). We find only one percolation fixed point $H = -\infty$, $K = 0$, $J = J_1$ which coincides with the random bond percolation fixed point already found in the ferromagnetic case.

This fixed point describes all the percolative transitions which occur in the (H, T) plane. The critical percolation lines are plotted in figure 5 for different values of p_B. All of them converge to the top of the phase boundary. This is rather intuitive, in fact at $T = 0$ $(1/K = 0)$ for $H/K < 1$ the ground state is ordered antiferromagnetically, which for the square lattice does not give any percolation. For $H/K > 1$ the ground state is ordered ferromagnetically which always gives percolation for all the values of $p_B > p_C$,

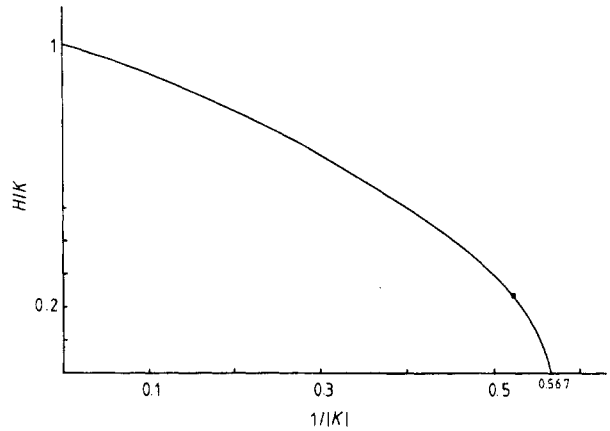

Figure 4. Antiferromagnetic phase boundary in the infinitesimal Migdal–Kadanoff renormalisation group approximation. Note that at $T=0$ $(1/K=0)$ and $T=T_C$ $(K=K_C)$ this approximation reproduces exact results. The dot on the phase boundary is a spurious fixed point due to the MK approximation.

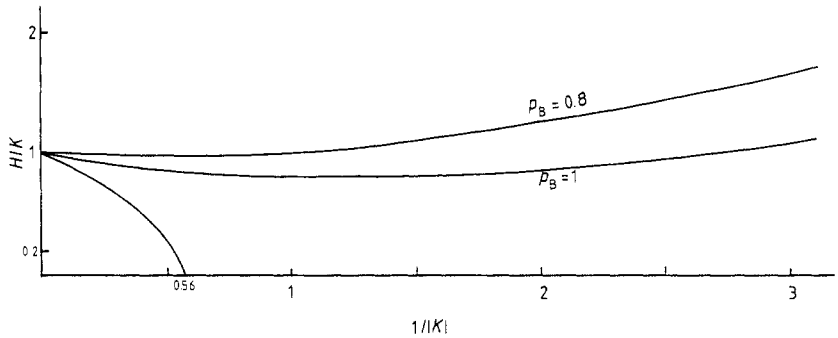

Figure 5. Antiferromagnetic phase boundary and the percolation lines for different values of p_B. All these lines end at the value $H/K=1$.

where p_C is the random bond percolation threshold. Of course for $p_B < p_C$ no percolation is allowed in the (H, T) plane.

Finally in figure 6 we give in the (H, K) plane an overall picture of the percolation line for both ferromagnetic and antiferromagnetic interactions.

In conclusion we have used a Hamiltonian formalism previously introduced to study site bond percolation with ferromagnetic correlation. The infinitesimal Migdal–Kadanoff renormalisation group has been employed to include also the antiferromagnetic interactions. This model might be of relevance in studying effects of solvent in polyfunctional condensation.

While in the ferromagnetic case for a range of values of p_B the cluster diverges at the Ising critical point, this never occurs in the antiferromagnetic case.

In order to define a droplet which diverges at the thermal critical point with the right exponent, as in the ferromagnetic case, one should define a cluster of particles and holes with ferromagnetic order. This problem is under investigation.

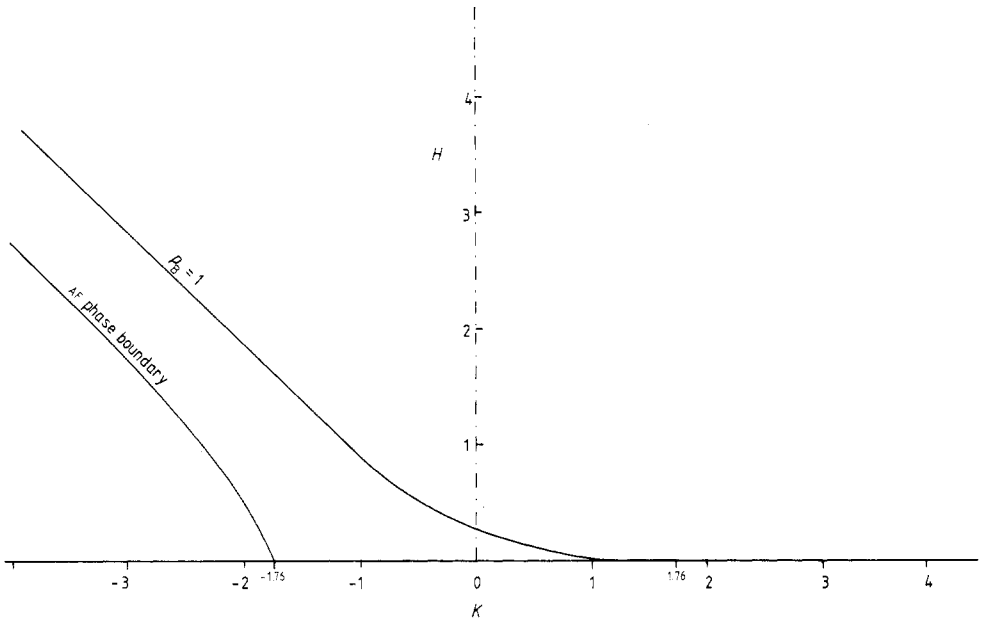

Figure 6. Antiferromagnetic phase boundary and the overall picture of the percolation line at $p_B = 1$ for negative and positive K, in the (H, K) plane. The percolation line is a monotonic increasing function of K.

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Appendix

Here for convenience we give an outline of the standard MK renormalisation procedure for a general Hamiltonian

$$-\beta\mathcal{H} = \sum_{\langle ij \rangle} \mathcal{H}(\mu_i, \mu_j)$$

where μ_i is a variable on site i which can assume t values and the sum is over all nearest neighbours on the square lattice made of $|V|$ sites. The partition function is

$$Z = \sum_{\{\mu\}_V} \prod_{ij} T(\mu_i, \mu_j)$$

where

$$T(\mu_i, \mu_j) = \exp(\mathcal{H}(\mu_i, \mu_j)).$$

and the sum is over all the configurations $\{\mu\}_V$ on the lattice V. The MK transformation can be done in many ways which will give the same results in the infinitesimal limit

(Nicoll 1979). We follow the procedure made of two steps schematically shown in figure 7 for b, the length rescaling factor, equal to 3.

Let V_3 denote the set of isolated vertices as shown in figure 7(b) and V_2 the set of vertices that are nearest neighbours of some vertex in V_3. Obviously $V = V_3 \cup V_2 \cup V_1$ where V_1 are the vertices in figure 7(c).

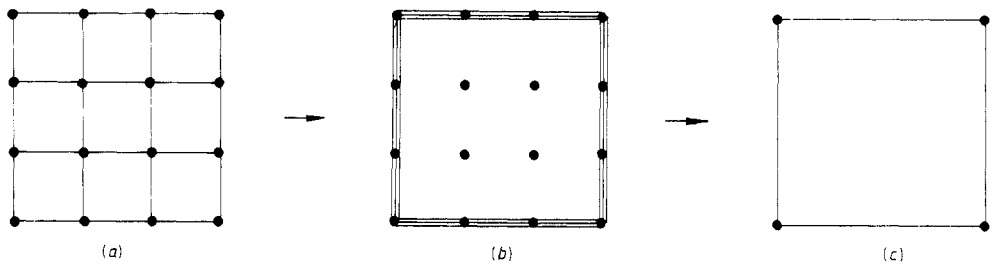

Figure 7. Successive steps of the Migdal–Kadanoff renormalisation group on the square lattice. (i) Bond moving $(a) \rightarrow (b)$, (ii) dedecoration $(b) \rightarrow (c)$.

(i) In the bond moving figure 7(a)→7(b) one requires that the overall energy is unchanged and that the interactions are only along the perimeter. The partition function is thus

$$Z = \sum_{\{\mu\}V} \exp \sum_{\langle ij\rangle \in V_1 \cup V_2} b\mathcal{H}(\mu_i, \mu_j) = \sum_{\{\mu\}V} \prod_{ij \in V_1 \cup V_2} T^b(\mu_i, \mu_j);$$

this can also be written

$$Z = t^{|V_3|} \sum_{\{\mu\}V_1} \prod_{ij \in V_1} T^b_{(\mu_i \mu_{h_1})} T^b_{(\mu_{h_1}, \mu_{h_2})} \ldots T^b_{(\mu_{h_b}, \mu_j)}$$

where $h_1 \ldots h_b$ is the sequence of sites connecting two nearest neighbour sites in V_1.

(ii) In the dedecoration transformation figure 7(b)→7(c) one eliminates the dependence on the sites V_2 by summing in the partition function over the variables in V_2; therefore we have

$$Z = t^{|V_3|} \sum_{\{\mu\}V_1} \prod_{ij \in V_1} T'_{(\mu_i, \mu_j)}$$

where

$$T'_{(\mu_i, \mu_j)} = \sum_{\{\mu\}V_2} T^b_{(\mu_i, \mu_{h_1})} \ldots T^b_{(\mu_{h_b}, \mu_j)}. \tag{A1}$$

The left-hand side can be considered as the elements of a matrix T' and the right-hand side as the elements of a product b times of a matrix T whose elements are given by $T^b_{(\mu_i, \mu_j)}$. Therefore symbolically equation (A1) can be written as

$$T' = T^b(T^b_{(\mu_i, \mu_j)}). \tag{A2}$$

As an example, for the lattice gas Hamiltonian

$$-\beta\mathcal{H}_{LG} = \sum_{\langle ij\rangle} K n_i n_j - \frac{\Delta}{C}(n_i + n_j) \tag{A3}$$

equation (A2) becomes

$$\begin{pmatrix} \exp(K'-2\Delta'/C+w_0') & \exp(-\Delta'/C+w_0') \\ \exp(-\Delta'/C+w_0') & \exp(w_0') \end{pmatrix} = \begin{pmatrix} \exp[b(K-2\Delta/C)] & \exp(-b\Delta/C) \\ \exp(-b\Delta/C) & 1 \end{pmatrix}$$

where w_0' is a constant. Diagonalising the above matrices we obtain the renormalisation equations (22*a*) and (22*b*).

Note that in (A3) the linear term has been treated as a bond.

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